

# A novel adaptive method for operator inclusions

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**Abstract** — A novel splitting algorithm for solving operator inclusion with the sum of the maximal monotone operator and the monotone Lipschitz continuous operator in the Banach space is proposed and studied. The proposed algorithm is an adaptive variant of the forward-reflected-backward algorithm, where the rule used to update the step size does not require knowledge of the Lipschitz constant of the operator. For operator inclusions in 2-uniformly convex and uniformly smooth Banach space, the theorem on the weak convergence of the method is proved.

**Keywords** — maximal monotone operator; operator inclusion; splitting algorithm; adaptability; 2-uniformly convex Banach space; uniformly smooth Banach space.

## I. INTRODUCTION

Let  $E$  be a real Banach space with its dual  $E^*$ . Consider the next operator inclusion problem:

$$\text{find } x \in E: 0 \in (A + B)x, \quad (1)$$

where  $A: E \rightarrow 2^{E^*}$  is multivalued maximal monotone operator,  $B: E \rightarrow E^*$  is monotone and Lipschitz continuous operator. Many actual problems can be written in the form of (1). Among them are variational inequalities and optimization problems arising in the field of optimal control, inverse problem theory, machine learning, image processing, operations research, and mathematical physics [1–3]. The most well-known and popular method for solving monotone operator inclusions (1) in Hilbert space is the forward-backward algorithm (FBA) [1, 4, 5]

$$x_{n+1} = J_{\lambda}^A(x_n - \lambda Bx_n),$$

where  $J_{\lambda}^A = (I + \lambda A)^{-1}$  is the operator resolvent,  $A: H \rightarrow 2^H$ ,  $\lambda > 0$ . Note that the FBA scheme includes well-known gradient method and proximal method as special case. For inverse strongly monotone (cocoercive) operators  $B: H \rightarrow H$  FBA method is weakly converging [1]. However, FBA may diverge for Lipschitz continuous monotone operators  $B$ . The

condition of the inverse strong monotonicity of the operator  $B$  is a rather strong assumption. To weaken it, Tseng [6] proposed the next modification of the FBA:

$$\begin{cases} y_n = J_{\lambda}^A(x_n - \lambda Bx_n), \\ x_{n+1} = y_n - \lambda(By_n - Bx_n), \end{cases}$$

where  $B: H \rightarrow H$  – monotone and Lipschitz continuous operator with constant  $L > 0$  and  $\lambda \in (0, L^{-1})$ . Further development of this idea led to the forward-reflected-backward algorithm [7]. Some progress has been achieved recently in the study of splitting algorithms for inclusions in Banach spaces [2, 8]. This is largely due to the wide involvement of theoretical results and designs of the geometry of Banach spaces [2, 9, 10]. Book [2] contains an extensive material on this topic.

The current work proposes and studies a new splitting algorithm for solving operator inclusion (1) in Banach space. The algorithm is an adaptive variant of the forward-reflected-backward algorithm, where the step update rule does not require knowledge of Lipschitz constant for operator  $B$ . The algorithm's advantage is only one computation at the iteration step of resolvent of maximal monotone operator  $A$  and value of operator  $B$ . The method weak convergence theorem is proved for operator inclusions in 2-uniformly convex and uniformly smooth Banach space.

## II. ALGORIHM

Let us recall several concepts and facts of the geometry of Banach spaces [2, 9–11], that are necessary for the formulation and proof of the results.

Let  $E$  be a real Banach space with norm  $\|\cdot\|$ ,  $E^*$  is the dual space for  $E$ . Let's denote norm in  $E^*$  as  $\|\cdot\|_*$ . Let  $S_E = \{x \in E: \|x\| = 1\}$ . Banach space is called strictly convex, if for all  $x, y \in S_E$  and  $x \neq y$  we have

$\left\|\frac{x+y}{2}\right\| < 1$ . The modulus of convexity of space  $E$  is defined as  $(\forall \varepsilon \in (0, 2])$

$$\delta_E(\varepsilon) = \inf \left\{ 1 - \left\|\frac{x+y}{2}\right\| : x, y \in S_E, \|x - y\| = \varepsilon \right\}.$$

A Banach space  $E$  is called uniformly convex, if  $\delta_E(\varepsilon) > 0$  for all  $\varepsilon \in (0, 2]$ . Banach space  $E$  is called 2-uniformly convex, if there exists such  $c > 0$  that  $\delta_E(\varepsilon) \geq c\varepsilon^2$  for all  $\varepsilon \in (0, 2]$ . Obviously, a 2-uniformly convex space is uniformly convex. It is known that the uniformly convex Banach space is reflexive [9]. A Banach space  $E$  is called smooth if the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \quad (2)$$

exists for all  $x, y \in S_E$ . A Banach space  $E$  is called uniformly smooth if the limit (2) exists uniformly over  $x, y \in S_E$ . There is a duality between the convexity and smoothness of the Banach space  $E$  and its dual space  $E^*$  [26, 27]. It is known that Hilbert spaces and spaces  $L_p$  ( $1 < p \leq 2$ ) are 2-uniformly convex and uniformly smooth (spaces  $L_p$  are uniformly smooth for  $p \in (1, \infty)$ ) [10]. Also recall [1, 11] that a multivalued operator  $A: E \rightarrow 2^{E^*}$  is called monotone if  $\forall x, y \in E$   $\langle u - v, x - y \rangle \geq 0 \quad \forall u \in Ax, v \in Ay$ . A monotone operator  $A: E \rightarrow 2^{E^*}$  is called maximal monotone if for any monotone operator  $B: E \rightarrow 2^{E^*}$  we have that  $\Gamma(A) \subseteq \Gamma(B)$  implies  $\Gamma(A) = \Gamma(B)$ , where  $\Gamma(A)$  is a graph of  $A$  [1]. It is known that if  $A: E \rightarrow 2^{E^*}$  is maximal monotone operator,  $B: E \rightarrow E^*$  is Lipschitz continuous monotone operator, then  $A + B$  is maximal monotone operator. Let us also recall [1] that operator  $A: E \rightarrow E^*$  is called inverse strongly monotone (cocoercive) if there exists such a number  $\alpha > 0$  (the constant of inverse strong monotonicity) that  $\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2$ . Inverse strongly monotone operator is Lipschitz continuous, but not every Lipschitz continuous operator is inverse strongly monotone. Multivalued operator  $J: E \rightarrow 2^{E^*}$ , which acts as

$$Jx = \left\{ x^* \in E^* : \langle x^*, x \rangle = \|x\|^2 = \|x^*\|^2 \right\},$$

is called normalized duality mapping. It is known [9, 10] that: if space  $E$  is smooth then operator  $J$  is single-valued; if space  $E$  is strongly convex then operator  $J$  is injective and strongly monotone; if space  $E$  is reflexive then operator  $J$  is surjective; if space  $E$  is uniformly smooth then operator  $J$  is uniformly continuous on bounded subsets of  $E$ . For a Hilbert

space  $J = I$ . Explicit form of operator  $J$  for Banach spaces  $\ell_p$ ,  $L_p$ , and  $W_p^m$  ( $p \in (1, \infty)$ ) is provided in [9–11]. Let  $E$  be a reflexive, strictly convex and smooth Banach space. The maximal monotonicity of operator  $A: E \rightarrow 2^{E^*}$  is equivalent to equality  $R(J + \lambda A) = E^*$  for all  $\lambda > 0$ . For maximal monotone operator  $A: E \rightarrow 2^{E^*}$  and  $\lambda > 0$  resolvent  $J_\lambda^A: E \rightarrow E$  is defined as follows

$$J_\lambda^A x = (J + \lambda A)^{-1} Jx, \quad x \in E,$$

where  $J$  is normalized duality mapping from  $E$  to  $E^*$ . It is known that

$$A^{-1}0 = F(J_\lambda^A) = \{x \in E : J_\lambda^A x = x\} \quad \forall \lambda > 0.$$

Let  $E$  be a smooth Banach space. Let's consider the functional introduced by Yakov Alber [11]

$$\phi(x, y) = \|x\|^2 - 2\langle Jy, x \rangle + \|y\|^2 \quad \forall x, y \in E.$$

If the space  $E$  is strictly convex, then for  $x, y \in E$  we have  $\phi(x, y) = 0 \Leftrightarrow x = y$ .

**Lemma 1 ([12, 13]).** Let  $E$  be a 2-uniformly convex and smooth Banach space. Then for some  $\mu \geq 1$  the next inequality holds:

$$\phi(x, y) \geq \frac{1}{\mu} \|x - y\|^2 \quad \forall x, y \in E.$$

For Banach spaces  $\ell_p$ ,  $L_p$  and  $W_p^m$  ( $1 < p \leq 2$ ) we have  $\mu = \frac{1}{p-1}$  [14]. And for a Hilbert space inequality for Lemma 3 becomes identity.

Let  $E$  be a 2-uniformly convex and uniformly smooth Banach space. Let  $A$  be a multivalued operator acting from  $E$  into  $2^{E^*}$ , and  $B$  an operator acting from  $E$  into  $E^*$ . Consider the operator inclusion problem (1) and Assume that the following conditions are satisfied:  $A: E \rightarrow 2^{E^*}$  is a maximal monotone operator;  $B: E \rightarrow E^*$  is a monotone and Lipschitz continuous operator with Lipschitz constant  $L > 0$ ; set  $(A + B)^{-1}0$  is not empty. Operator inclusion (1) can be formulated as the problem of finding a fixed point:

$$\text{find } x \in E : x = J_\lambda^A \circ J^{-1}(Jx - \lambda Bx), \quad (3)$$

where  $\lambda > 0$ . Formulation (3) is useful because it contains an obvious algorithmic idea. Calculation scheme  $x_{n+1} = J_\lambda^A \circ J^{-1}(Jx_n - \lambda Bx_n)$  was studied in [8] for inverse strongly monotone operators  $B: E \rightarrow E^*$ . However, the scheme generally does not converge for Lipschitz continuous monotone operators. Let's use the idea of work [7] and consider modified scheme

$$x_{n+1} = J_\lambda^A \circ J^{-1}(Jx_n - \lambda Bx_n - \lambda(Bx_n - Bx_{n-1}))$$

with extrapolation term  $-\lambda(Bx_n - Bx_{n-1})$ , and let's use update rule for  $\lambda > 0$  like one from [15] to exclude explicit use of Lipschitz constant of operator  $B$ . We will assume that we know constant  $\mu$  from Lemma 1.

#### Algorithm 1

Choose some  $x_0 \in E$ ,  $x_1 \in E$ ,  $\tau \in (0, \frac{1}{2\mu})$  and

$\lambda_0, \lambda_1 > 0$ . Set  $n = 1$ .

1. Compute

$$x_{n+1} = J_{\lambda_n}^A \circ J^{-1} \left( Jx_n - \lambda_n Bx_n - \lambda_{n-1} (Bx_n - Bx_{n-1}) \right).$$

2. If  $x_{n-1} = x_n = x_{n+1}$ , then STOP and

$$x_n \in (A + B)^{-1} 0, \text{ else go to 3.}$$

3. Compute

$$\lambda_{n+1} = \begin{cases} \min \left\{ \lambda_n, \tau \frac{\|x_{n+1} - x_n\|}{\|Bx_{n+1} - Bx_n\|_*} \right\}, & \text{if } Bx_{n+1} \neq Bx_n, \\ \lambda_n, & \text{otherwise.} \end{cases}$$

Set  $n := n + 1$  and go to 1.

Sequence  $(\lambda_n)$  which is created by rule on step 3 is non-increasing and bounded from below by  $\min \{ \lambda_1, \tau L^{-1} \}$ . So, there exists  $\lim_{n \rightarrow \infty} \lambda_n > 0$ .

### III. MAIN RESULT

In begin this section, we state the inequality on which the proof of Algorithm 1 weak convergence is based.

**Lemma 2.** For the sequence  $(x_n)$ , generated by Algorithm 1 the following inequality holds:

$$\begin{aligned} \phi(z, x_{n+1}) + 2\lambda_n \langle Bx_n - Bx_{n+1}, x_{n+1} - z \rangle + \\ + \tau\mu \frac{\lambda_n}{\lambda_{n+1}} \phi(x_{n+1}, x_n) \leq \\ \leq \phi(z, x_n) + 2\lambda_{n-1} \langle Bx_{n-1} - Bx_n, x_n - z \rangle + \\ + \tau\mu \frac{\lambda_{n-1}}{\lambda_n} \phi(x_n, x_{n-1}) - \\ - \left( 1 - \tau\mu \frac{\lambda_{n-1}}{\lambda_n} - \tau\mu \frac{\lambda_n}{\lambda_{n+1}} \right) \phi(x_{n+1}, x_n), \end{aligned}$$

where  $z \in (A + B)^{-1} 0$ .

Let us formulate the main result.

**Theorem 1.** Let  $E$  be a 2-uniformly convex and uniformly smooth Banach space,  $A: E \rightarrow 2^{E^*}$  be a maximal monotone operator,  $B: E \rightarrow E^*$  be a monotone and Lipschitz continuous operator,  $(A + B)^{-1} 0 \neq \emptyset$ . Suppose that the normalized duality map  $J$  is sequentially weakly continuous. Then sequence  $(x_n)$  generated by Algorithm 1 converges weakly to some point  $z \in (A + B)^{-1} 0$ .

### IV. CONCLUSIONS

In this paper new splitting algorithm for solving an operator inclusion with the sum of a maximal monotone operator and a monotone Lipschitz continuous operator in a Banach space is proposed and studied. The algorithm is an adaptive variant of the forward-reflected-backward algorithm of Malitsky–Tam, where the used rule for updating the step size does not require knowledge of the Lipschitz constant of operator  $B$ . An attractive feature of the algorithm is only one computation of the resolvent of the maximal monotone operator  $A$  and the value of the monotone Lipschitz continuous operator  $B$  at the iteration step. Theorem on the weak convergence of the method is proved for operator inclusions in a 2-uniformly convex and uniformly smooth Banach space. An interesting challenge for the future is the development of a strongly convergent modification of the proposed algorithm.

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