M-modification of the "Pyramidal" Method of Data Extrapolation

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Abstract— **In this work certain generalizations of the classical derivative of a differentiable function are introduced. The method of time series extrapolation is proposed on the basis of corresponding generalizations. In the basis of this method is the analysis of separated differences. It is proposed procedure of the corresponding differences modification and finding of such order, for which it is possible to find in the certain sense the best forecast value. Then the value of the output function at a point that lies outside the interpolation interval is based on the found predictive value for the separated differences using a special computational procedure.**

Keywords— extrapolation; forecast; divided differences; interpolation, time series recognition problems.

I. Introduction

Today the series of quantitative and qualitative approaches of forecasting are known [1,7]. The most common methods of short time-series forecasting are the extrapolation methods. There is a number of problems dealing with the study of small time series. For example, in the absence of additional information is often impossible to conclude that process is determinate or indeterminate, which significantly effects on the model construction. In determinate case is obvious that for any set of points there are many curves that pass through them or something to bring, and it is difficult to argue that one curve (model) is precisely the law that comprehensively describes the phenomenon and will effectively predict its behavior in the future.

The hypothesis, which corresponds to the observation point, is effective if it is appropriate to predict with reasonable accuracy the process starting from any step of observations. In practice, the finding an effective hypothesis is impossible.

In this paper we analyze a special method of extrapolation which was presented in [1] and propose a special extension of this method. The "piramidal" method is based on the property of the rows of modified finite differences that provide the best cubic approximation in the range of convexity. Numerical results show significant advantages of the proposed method in comparison with approaches to extrapolate, based on the using of polynomials, including Newton's extrapolation.

II. $\mu\lambda$ - Derivatives and Their Property

At the heart of our idea of finite differences generalizing is the following modification of the notion of a derivative, which we have called a $\mu\lambda$ -derivative. Now we shall give the following definition.

Definition. The boundary

$$f^{\partial}(x)=\lim_{\Delta\to 0}\frac{f(x+\Delta)-\mu^*(\Delta)f(x)-\lambda^*(\Delta)}{\Delta}$$

is called $\mu\lambda$ - derivative of some differential function $f(x)$ if $\mu^*(\Delta)$, $\lambda^*(\Delta)$ are solutions of the problem:

$$\int_\Omega (f(x+\Delta)-\mu^*(\Delta)f(x)-\lambda^*(\Delta))^2\,p(dx)\to\min_{\lambda^*,\mu^*} \qquad (1)$$

Obviously, that

$$\lambda^*(\Delta)=(\int_\Omega f(x+\Delta)p(dx)-\mu^*(\Delta)\int_\Omega f(x)p(dx))/\int_\Omega p(dx),$$

$$\mu^*(\Delta)=(\int_\Omega f(x+\Delta)p(dx)\int_\Omega f(x)p(dx)/\int_\Omega p(dx)-$$
$$-\int_\Omega f(x+\Delta)f(x)p(dx))/ \qquad (2)$$
$$/((\int_\Omega f(x)p(dx))^2/\int_\Omega p(dx)-\int_\Omega f^2(x)dx).$$

Note that the Lebesgue integral in the definition can be replaced by the defined Riemann integral at some finite interval. Moreover, in some problems condition (1) may be absent, then $\mu^*(\Delta),\lambda^*(\Delta)$ are parameters. Consider the examples.

Example 1. Let consider linear function $f(x)=ax+b$. Then $\mu^*(\Delta)=1, \lambda^*(\Delta)=a\Delta$,

$$\int_c^d (a(x+\Delta)+b-\mu^*(\Delta)(ax+b)-\lambda^*(\Delta))^2 dx = 0.$$

We obtaine:

$$\lim_{\Delta\to 0}\frac{a(x+\Delta)+b-ax-b-a\Delta}{\Delta}=0.$$

Example2. Let $f(x)=e^x$. Thus

$$\int_\Omega (f(x+\Delta)-\mu^*(\Delta)f(x)-\lambda^*(\Delta))^2 p(dx) = $$
$$=\int_c^d (e^{x+\Delta}-\mu^*(\Delta)e^x-\lambda^*(\Delta))^2 dx,$$

$$\mu^*(\Delta)=e^\Delta, \lambda^*(\Delta)=0, \int_c^d (e^{x+\Delta}-\mu^*(\Delta)e^x-\lambda^*(\Delta))^2 dx = 0.$$

Than

$$f^\partial(x)=\lim_{\Delta\to 0}\frac{f(x+\Delta)-\mu^*(\Delta)f(x)-\lambda^*(\Delta)}{\Delta}=$$
$$=\lim_{\Delta\to 0}\frac{e^{x+\Delta}-\mu^*(\Delta)e^x-\lambda^*(\Delta)}{\Delta}=0.$$

Example 3. Let $f(x)=x^m$.

Using (2) we obtaine:

$$\mu^*(\Delta)=1+m\Delta C+O(\Delta^2),$$
$$\lambda^*(\Delta)=(\int_\Omega f(x+\Delta)p(dx)-\mu^*(\Delta)\int_\Omega f(x)p(dx))/\int_\Omega p(dx)=$$
$$=(\int_c^d (x+\Delta)^m dx-(1+m\Delta C)\int_c^d (x)^m dx)/(d-c)=$$
$$=m\Delta(\int_c^d (x)^{m-1}dx - C\int_c^d (x)^m dx)/(d-c)+O(\Delta^2).$$

Therefore,

$$f^\partial(x)=\lim_{\Delta\to 0}\frac{(x+\Delta)^m-\mu^*(\Delta)x^m-\lambda^*(\Delta)}{\Delta}=$$
$$=mx^{m-1}(1-Cx)-mC_1,$$
$$C=(\int_c^d x^{m-1}dx\int_c^d (x)^m dx/(d-c)-$$
$$-\int_c^d x^{m-1}(x)^m dx)/((\int_c^d (x)^m dx)^2/(d-c)-\int_c^d (x)^{2m}dx)$$
$$C_1=(\int_c^d (x)^{m-1}dx - C\int_c^d (x)^m dx)/(d-c).$$

The considered properties of the our derivative show that, unlike the classical derivative, in the presence of condition (1) it “nullifies” the values of functions having exponential growth order or linear functions. Therefore, it is obvious that using of the difference analogues of such a derivative will significantly improve the prediction methods based on the using of finite differences or certain modifications. In particular, using our approuch improved the pyramidal method of extrapolation proposed in [1-3].

III. Generelization of the piramidal method

Suppose we have the value of some function $f_1, f_2,..., f_n$ defined in points $x_1, x_2,..., x_n$ respectively. The classic problem of extrapolation is to estimate the value of this function at the point $x > x_n$.

Further generalization of the "pyramidal" method described in [1-4] is based on the application of the corresponding finite-difference generalizations. Consider the midpoints $x_i^c = (x_i + x_{i+1})/2$ and the following generalizations of finite differences:

$$\Delta^j f_i = \frac{\Delta^{j-1} f_{i+1} - \mu_{j-1}\Delta^{j-1} f_i - \lambda_{j-1}}{r_i^j - l_i^j}, \tag{3}$$

where

$$r_i^j = \begin{cases} x^c_{i+j/2}, j=2k, \\ x_{i+[j/2]+1}, j=2k+1, \end{cases} \quad l_i^j = \begin{cases} x^c_{i+j/2-1}, j=2k, \\ x_{i+[j/2]}, j=2k+1, \end{cases}$$

$$\Delta^j f_i^c = \frac{\Delta^{j-1} f_{i+1}^c - \mu_{j-1}\Delta^{j-1} f_i^c - \lambda_{j-1}}{\hat{r}_i^j - l_i^j}, \tag{4}$$

where

$$\hat{r}_i^j = \begin{cases} x_{i+j/2+1}, j=2k, \\ x^c_{i+[j/2]+1}, j=2k+1, \end{cases} \quad \hat{l}_i^j = \begin{cases} x_{i+j/2}, j=2k, \\ x^c_{i+[j/2]}, j=2k+1, \end{cases}$$

$$\tilde{\Delta}^i f^c_{n-i} = 2\frac{\left(\Delta^{i-2} f_{n-i+2} - \mu_{i-2}\Delta^{i-2} f^c_{n-i+1} - \lambda_{i-2}\right)}{(r_i - c_i)(r_i - l_i)} - $$
$$-2\mu_{i-1}\frac{\left(\Delta^{i-2} f^c_{n-i+1} - \mu_{i-2}\Delta^{i-2} f_{n-i+1} - \lambda_{i-2}\right)}{(c_i - l_i)(r_i - l_i)} - \lambda_{i-2}, \tag{5}$$

where

$$r_i = \begin{cases} x_{n-\frac{i}{2}+1}, i=2k, \\ x^c_{n-\left[\frac{i}{2}\right]}, i=2k+1, \end{cases}$$

$$rr_i = \begin{cases} x^c_{n-\frac{i}{2}+1}, i=2k, \\ x_{n-\left[\frac{i}{2}\right]+1}, i=2k+1, \end{cases} \quad c_i = \begin{cases} x^c_{n-\frac{i}{2}}, i=2k, \\ x_{n-\left[\frac{i}{2}\right]}, i=2k+1, \end{cases}$$

$$l_i = \begin{cases} x_{n-\frac{i}{2}}, i=2k, \\ x^c_{n-\left[\frac{i}{2}\right]-1}, i=2k+1, \end{cases} \quad ll_i = \begin{cases} x^c_{n-\frac{i}{2}-1}, i=2k, \\ x_{n-\left[\frac{i}{2}\right]-1}, i=2k+1. \end{cases},$$

$i=\overline{2, n-1}$.

Suppose that:

$$\widetilde{\Delta}^i f^c_{n-i} = \Delta^i f^c_{n-i} \quad (6)$$

Then the procedure for finding the unknown value of a function at the point x^c_n is defined as:

$$\Delta^{j-1} f^c_{n-j+1} = \mu_{j-1}\Delta^{j-1} f^c_{n-j} + \Delta^j f^c_{n-j}\left(\hat{r}^j_{n-j} - \hat{l}^j_{n-j}\right) + \lambda_{j-1},$$

$$j = \overline{i,1}. \quad (7)$$

The essence of the further considerations is to determine such a row of the modified finite difference table, in which the relation (6) is executed with minimal error. As was shown in [1], the condition that the points $\left(c_i, \widetilde{\Delta}^i f^c_{n-i}\right)$, $\left(l_i, \widetilde{\Delta}^i f^c_{n-i-1}\right)$, $\left(r_i, \widetilde{\Delta}^i f^c_{n-i+1}\right)$ lie on a straight line is a necessary and sufficient condition for the fulfillment of (6). Obviously, the latter condition will be satisfied (with an accuracy corresponding to the accuracy of the approximation of the second derivative by finite differences, if the interval of the curve passing through the corresponding points is a cubic polynomial. If the parameters $\mu^*(\Delta), \lambda^*(\Delta)$ are determined from condition (1), the corresponding and the sufficient condition for executing (6) will be different.

Having made simple but rather cumbersome transformations, it can be shown that condition (6) is written as:

$$\left(\frac{\left(\Delta^{i-2} f_{n-i+2} - \Delta^{i-2} f^c_{n-i+1}\right)}{(r_i - c_i)} - \frac{\left(\Delta^{i-2} f^c_{n-i+1} - \Delta^{i-2} f_{n-i+1}\right)}{(c_i - l_i)}\right) -$$

$$-\left(\frac{\left(\Delta^{i-2} f^c{}_{n-i+2} - \Delta^{i-2} f_{n-i+2}\right)}{(rr_i - r_i)} - \frac{\left(\Delta^{i-2} f_{n-i+2} - \Delta^{i-2} f^c{}_{n-i+1}\right)}{(r_i - c_i)}\right) *$$

$$* \frac{rr_i - r_i}{rr_i - c_i} -$$

$$-\left(\frac{\left(\Delta^{i-2} f^c_{n-i+1} - \Delta^{i-2} f_{n-i+1}\right)}{(c_i - l_i)} - \frac{\left(\Delta^{i-2} f_{n-i+1} - \Delta^{i-2} f^c{}_{n-i}\right)}{(l_i - ll_i)}\right)\frac{l_i - ll_i}{c_i - ll_i} =$$

$$= (1-\mu_{i-1})\frac{\left(\Delta^{i-2} f^c_{n-i+1} - \Delta^{i-2} f_{n-i+1}\right)}{(c_i - l_i)}\left(\frac{c_i - l_i}{c_i - ll_i} - 2\right) +$$

$$+ (1-\mu_{i-1})\frac{\left(\Delta^{i-2} f_{n-i+1} - \Delta^{i-2} f^c{}_{n-i}\right)}{(l_i - ll_i)}\frac{l_i - ll_i}{c_i - ll_i} +$$

$$+ \frac{(1-\mu_{i-2})\left(\Delta^{i-2} f^c_{n-i+1} - \lambda_{i-2}\right)}{(r_i - c_i)}\left(\frac{r_i - c_i}{rr_i - c_i} - 2\right) - \quad (8)$$

$$-\mu_{i-1}\frac{(1-\mu_{i-2})\Delta^{i-2} f^c_{n-i} - \lambda_{i-2}}{(c_i - ll_i)} +$$

$$+ 2\mu_{i-1}\frac{(1-\mu_{i-2})\Delta^{i-2} f_{n-i+1} - \lambda_{i-2}}{(c_i - l_i)} + \lambda_{i-1}$$

Obviously that when $\mu_i = 1, \lambda_i = 0, i = \overline{1,n}$, the last relation turns into the condition of belonging to one straight point $\left(c_i, \widetilde{\Delta}^i f^c_{n-i}\right)$, $\left(l_i, \widetilde{\Delta}^i f^c_{n-i-1}\right)$, $\left(r_i, \widetilde{\Delta}^i f^c_{n-i+1}\right)$, which was considered in [1]. Note that relation (7) cannot be used directly to construct a prediction because $\Delta^{i-2} f_{n-i+2}$ contains an unknown predictive value. To construct a constructive condition for the relation (7) we consider a continuous analog (7) and obtain the equation:

$$f''(c_i) - f''(r_i)\frac{rr_i - r_i}{rr_i - c_i} - f''(l_i)\frac{l_i - ll_i}{c_i - ll_i} =$$

$$= (1-\mu_{i-1}) f'((c_i + l_i)/2)\left(\frac{c_i - l_i}{c_i - ll_i} - 2\right) +$$

$$+ (1-\mu_{i-1}) f'((l_i + ll_i)/2)\frac{l_i - ll_i}{c_i - ll_i} +$$

$$\frac{(1-\mu_{i-2})\left(f(c_i) - \lambda_{i-2}\right)}{(r_i - c_i)}\left(\frac{r_i - c_i}{rr_i - c_i} - 2\right) -$$

$$-\mu_{i-1}\frac{(1-\mu_{i-2}) f(ll_i) - \lambda_{i-2}}{(c_i - ll_i)} +$$

$$+ 2\mu_{i-1}\frac{(1-\mu_{i-2})\Delta^{i-2} f_{n-i+1} - \lambda_{i-2}}{(c_i - l_i)} + \lambda_{i-1}$$

For a uniform grid, we obtain a differential equation with a delay of the form:

$$f''(x) - f''(x+\Delta)\frac{1}{2} - f''(lx - \Delta)\frac{1}{2} =$$

$$= -\frac{3}{2}(1-\mu_{i-1}) f'((x - \Delta/2)/2) + \frac{1}{2}(1-\mu_{i-1}) f'(x - 3\Delta/2) -$$

$$-\frac{3}{2}\frac{(1-\mu_{i-2})\left(f(x) - \lambda_{i-2}\right)}{\Delta} - \mu_{i-1}\frac{(1-\mu_{i-2}) f(x - 2\Delta) - \lambda_{i-2}}{2\Delta} +$$

$$+ 2\mu_{i-1}\frac{(1-\mu_{i-2}) f(x-\Delta) - \lambda_{i-2}}{\Delta} + \lambda_{i-1}$$

We will find its partial solution in the form: $f(x) = a + e^{bx}$. Substituting in (9), we obtain a system of equations for finding unknown parameters:

$$3(\mu_{i-1} - 1)\frac{1-\mu_{i-2}}{2\Delta}a = 3(\mu_{i-1} - 1)\frac{\lambda_{i-2}}{2\Delta} - \lambda_{i-1},$$

$$b^2 - \frac{1}{2}b^2 e^{b\Delta} - \frac{1}{2}b^2 e^{-b\Delta} = \frac{3}{2}(\mu_{i-1} - 1)be^{-b\Delta/2} + \frac{1}{2}(1-\mu_{i-1})be^{-3b\Delta/2} -$$

$$-\frac{3}{2}(1-\mu_{i-2})/\Delta - \mu_{i-1}(1-\mu_{i-2})e^{-2b\Delta}/(2\Delta) +$$

$$+ 2\mu_{i-1}(1-\mu_{i-2})e^{-b\Delta}/\Delta.$$

The transcendental equation for an unknown parameter b has a solution that can be found approximate. For example, we can get values $\Delta = 1, \mu_{i-1} = 1.5, \mu_{i-2} = 6.4$. Thus, we see that to satisfy condition (5) it is sufficient that the points $\left(c_i, \widetilde{\Delta}^i f^c_{n-i}\right)$, $\left(l_i, \widetilde{\Delta}^i f^c_{n-i-1}\right)$, $\left(r_i, \widetilde{\Delta}^i f^c_{n-i+1}\right)$ lie on the curve determined by the function of the form: $f(x) = a + e^{bx}$. The same condition for the classical differences, as mentioned above, was related to the cubic polynomial.

So, we can offer the following extrapolation algorithm:

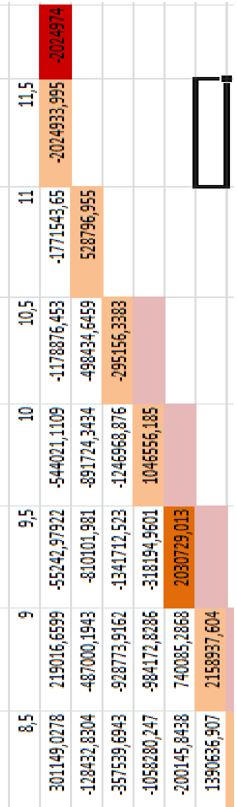
1. A table of modified finite differences is built.
2. The function is interpolated at midpoints, and the finite difference table is supplemented by averages.
3. For each row of the finite difference table, conditions (5) are checked, in particular, under a number of sufficient conditions, and a row for which the error of relation (5) is minimal is determined
4. The estimated value $\tilde{\Delta}^{i^*} f^{c}_{n-i^*}$ is calculated.
5. The predicted value of the function is given by formulas (6).

IV . Numerical results

Consider an example application of the approach described above. Let us have the test function $f(x)=x^6\sin(x)$ defined in points 1, 1.5, 2, 2.5, 3, … , 11. A similar example was considered in [1]. The pyramidal method in [1] allowed to obtain a predicted value of the function equal to -2017907,745, with the exact value of -2024974,077. Consider now a modification of the method using-derivatives. A fragment of the table of modified split differences is shown in Fig. 2. As you can see, replacing the classic differences with the differences in the first and second rows of the split difference table gives you a prediction value at 11.5, which is -2024933,995 for the corresponding test example.

Using our generalization of finite differences allowe to obtain a modification of the extrapolation method previously proposed by the authors. This modification has a significant advantage in cases where the number of observations for which the projected value is constructed has an exponential growth pattern. The numerical results show the significant advantages of the proposed method over the extrapolation approaches based on the use of polynomials, in particular the Newton polynomial of the second kind.

The proposed methodology is of a general nature and can be used to extrapolate time series into arbitrary fields of research, in particular when constructing short-term forecasts of economic dynamics series.



8,5	9	9,5	10	10,5	11	11,5	
301149,0278	219016,6599	-55242,97922	-544021,1109	-1178876,453	-1771543,65	-2024933,995	-2024974
-128432,8304	-487000,1943	-810101,981	-891724,3434	-498434,6459	528796,955		
-357539,6943	-928773,9162	-1341712,523	-1246958,876	-295156,3383			
-1058280,247	-984172,8286	-318194,9601	1046556,185				
-200145,8438	740085,2868	2030729,013					
1390636,907	2158937,604						

Figure1. Part of the table of finite differencis

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