Analysis of Stability of One Class of Numerical Methods of Second Order

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Abstract— **An iterative approach to construction numerical methods second order based on the method of Linigur-Whillaby, that have minimal error of discretization is proposed. The essence of the approach in defining amendments to the explicit and implicit Euler's method at the time when their contributions are equivalent. The stability conditions for this class of methods are obtained by the example of conservative systems. The confirmed improving of characteristics time and accuracy in the process of computation in quartz generators 9 order.**

Keywords— numerical methods; discretization; stability conditions, quartz generators; auto-oscillation system; discrete formula; error of discretization.

I. Introduction

In work a new numerical method is offered for forming of the discrete equations during conducting of analysis of systems oscillation both autonomous, and with external forces. A discrete method is built such, that for obtain the next discrete point of state variables on integral curves, that answers the continuous model of process oscillation, need enough limited size of integration step at condition that a previous discrete point is found exactly. The stability conditions for this class of methods are obtained by the example of conservative systems.

Expedience of application of the developed method is shown to the analysis of quartz generators with high-Q quality [1, 2] and for generators with the long transitional processes [4-8, 13], that work in the regime of discrete time and for objects with difficult dynamic nature of such, as systems of recognition of computers users that based on the probabilistic approach [14].

II. APPROACH TO CREATING OF METHOD

Among a plenty of implicit numerical methods, which found wide application at the analysis of the hard systems (implicit methods Eylera, Rounge-Coutta, many-stepping methods Adamsa-Moultona, Gira, formulas of differentiation backwards, that based on application of Legandra polynomials [3, 9-14]) there are methods with the variable step of calculations, that enable to realize by main advantage of implicit numerical methods in comparative with explicit numerical methods – changing of step in wide scopes at finding of point of phase space in areas with a different rate of movement. Methods higher than the second order of complication, practically, are not used. This is due with growth of calculable complication of algorithms, and from the second side that the methods of the second order allow to study dynamics of the system of any nature and difficulty of conduct.

The analysis of literary sources confirms, in considerable part of the applied researches of dynamics of the hard systems is used the method of trapezoids, discrete formula of which has a kind:

$$x[n+1] = x[n] + \frac{h}{2} \cdot \left(f[x[n], t_n] + f[x[n+1], t_{n+1}] \right), \quad (1)$$

where x[n], $x[n+1]$ – value of state variables in the n and n+1-points of discretization; $f[x[n],t_n]$ $f[x[n+1],t_{n+1}]$ – value of right parts of the differential equation of the explore system, which written in normalized form Kashy.

A formula (1) is averaging amendments for implicit and explicit of methods Euler for calculate the next point of discretization, that determined in the moment of time on half size step integration. She coincides with the method of Adamsa-Bashforta first order, which near to the implicit method Heyna, in which the exact meaning of derivative at the end of interval of calculations is transferable on the close value of derivative, calculated in the middle of step of integration.

With the purpose of increase of exactness of method of trapezoids expediently to modify him thus, to provide the exact hit in the n+1 point of discrete during conducting of integration on condition that the values of state variables in a n point are calculated exactly. This it is possible to attain, if in (1) the step of integration is variable, and derivative (right parts of the differential equation) calculated in such moment on interval of h, that amendments of implicit method Eylera (second element of right part of formula (1)) and explicit method Eylera (third element of right part of formula (1)) are equal. Thus, we build a discrete method in a kind

$$x[n+1] = x[n] + h \cdot \left(\mu \cdot f[x[n], t_n] + (1-\mu) \cdot f[x[n+1], t_{n+1}] \right), \quad (2)$$

here

$$0 < \mu < 1.$$

Based on equation of two tangents, conducted on one curve in different moments of time, we can to define the value of parameter in that moment when amendments to implicit and explicit methods Eylera in a formula (2) are equivalent. Thus, from condition

$$\mu \cdot f[x[n],t_n] = (1-\mu) \cdot f[x[n+1],t_{n+1}] ,$$

we get that

$$\mu = \frac{f[x[n+1],t_{n+1}]}{(f[x[n+1],t_{n+1}] + f[x[n],t_n])} .$$

Consequently, we get a numerical discrete formula for construction of discrete models of the systems with different areas of slow and rapid of motions:

$$x[n+1] = x[n] + h \cdot \frac{2 \cdot f[x[n+1],t_{n+1}] \cdot f[x[n],t_n]}{(f[x[n+1],t_{n+1}] + f[x[n,t_n])} . \quad (3)$$

The results of computer design confirm a right to existence of such method what gives adequate recreation of descriptions of the set mode of oscillatory circuits without the losses, of generators with high quality, quartz generators with protracted transitional processes.

The negative in the resulted method, is comparatively with the method of trapezoid, in growth of algorithmic and computational complexity (almost in two times). Advantage in that at the comparable accuracy with the method of trapezoids on one step of integration, the developed method substantially extends the region of convergence of calculable procedures. In result at the calculation of generator with the protracted transitional processes, which are described by the differential equation of high orders, is necessity of application of methods of acceleration of search of the steady modes [4-7]. For providing of convergence of process of deductions, at the use of the discrete method (3), is possible to do without such procedure. Application of method (3) provides winning in time of calculation in 3 – 10 times in comparatively with the method of trapezoid and depends from dimension of equations that describe model generator and it's of quality.

As for the construction of the formula (3) the contribution of each of the Euler method does not exceed half the distance between x_n and x_{n+1}, thus the method (3) provides guaranteed limits on the error of discretization at each step and ensures its with positive sign.

Further, to facilitate entry new discrete formulas of the function at discrete point $x(t_n)$ will record as f_n.

III. Iteration Approach to Minimization Error of Discretization

The analysis of error calculation on the example of the conservative model of second order (on example oscillation circuit without losses) confirmed that the error of calculation method (3) is proportional to $h^2/24$, as in the method of trapezoids, but has the opposite sign and twice more the absolute value. After used for the first half step of formula (3), and then formula (1), obtain discrete formula:

$$x_{n+1} = x_n + \frac{h \cdot f_n \cdot f_{n+1}}{(f_n + f_{n+1})} + \frac{h}{4}(f_n + f_{n+1}), \quad (4)$$

named as a difference combination of the first kind (C1K). The error of calculation when using (4) to the conservative system second order was two times lower compared with the method of trapezoids and opposite in sign. Now, after averaging (1) and (4) we obtain difference combination of the second kind (C2K):

$$x_{n+1} = x_n + \frac{h \cdot f_n \cdot f_{n+1}}{2 \cdot (f_n + f_{n+1})} + \frac{3 \cdot h}{8}(f_n + f_{n+1}) . \quad (5)$$

As shown the results of the analysis error calculation method (5) on the model without loss, then it was 4 times less than the error of the method of trapezoids and two times less than the error of method (4). This error in the C2K coincides with the sign error in the method of trapezoids and opposite with the sign to the error C1K. Thus, we can expect further decrease of the error calculation of methods (5) and (6), which leads to a difference combination of third kind C3K):

$$x_{n+1} = x_n + \frac{3 \cdot h \cdot f_n \cdot f_{n+1}}{4 \cdot (f_n + f_{n+1})} + \frac{5 \cdot h}{16}(f_n + f_{n+1}) . \quad (6)$$

Note that consider a combination of (6) to (4) inappropriate (although she has a right to exist), because (5) has in 4 times smaller of error calculation, compared to (4). In addition the signs of errors in (4) and (6) coincide. The proposed combination of difference schemes constructed so, that in combinations of odd genus (C1K, C3K) is more significant contribution to the second member in the derived formulas, compared with the third, but in even combinations of type (C2K) these deposits are almost equivalents. This construction provides a change of sign of error in obtaining the new combination. Thus, we can construct a метод of any higher order, which will provide up to the members of the second order smallness arbitrarily small error. After arithmetic averaging (5) and (6) we arrive at the difference scheme fourth generation (C4K):

$$x_{n+1} = x_n + \frac{5 \cdot h \cdot f_n \cdot f_{n+1}}{8 \cdot (f_n + f_{n+1})} + \frac{11 \cdot h}{32}(f_n + f_{n+1}) . \quad (7)$$

Analyzing formula (4) - (7), for k-th step at using on half-step steam combination and half step odd, we obtain the difference scheme for the combination of *k*-th family (CKK):

$$x_{n+1} = x_n + \frac{a_k \cdot h \cdot f_n \cdot f_{n+1}}{(f_n + f_{n+1})} + a_{k+1} \cdot h \cdot (f_n + f_{n+1}), \quad (8)$$

where $a_k - \frac{2^k - (-1)^k}{3 \cdot 2^{k-1}}$; $a_{k+1} = \frac{2^{k+1} + (-1)^k}{3 \cdot 2^{k+1}}$.

Obviously, with increasing k the coefficients a_k and a_{k+1} in decrease, leading to a decrease of error calculation.

Thus, the error of calculation any *k*-th combinations can be estimated by the formula:

$$\delta = \frac{(-1)^k}{2^{k+1}} , \quad (9)$$

that confirming the analysis of conservative second-order systems, systems of high order and with high quality factor.

In order to minimize errors in (8) we will make the passage to the limit, directing *k* to infinity. Thus, we get the difference scheme (10). In her with accuracy to members of the second-order of smallness the errors are absent:

$$x_{n+1} = x_n + \frac{2 \cdot h \cdot f_n \cdot f_{n+1}}{3 \cdot (f_n + f_{n+1})} + \frac{1}{3} \cdot h \cdot (f_n + f_{n+1}). \quad (10)$$

The conclusion that there is no error of difference scheme (10) that is the modification of method trapezoid (MMT) follows from formula (9), if in its parameter k goes to infinity.

The obtained results of error evaluation for difference schemes (5 - (8) and (10) confirmed at the modeling steady-state regimes of the generator Van der Pol [1, 4, 7]. In that model is introduced cubic nonlinearity to delay the transition process. The above analytical error of calculations (9) fully confirmed our results of computer simulation for different values of the parameters of the generator Van der Pol.

The analysis high quality generator that described by a system of differential equations of order 18 at the choice 200 points in every steep, when we used method trapezoids (1) or method (10), that provide normal mode search steady state. At that the use of (1) requires twice more time consuming than (10). With increasing step twice, trapezoid method does not provide the convergence process of deduction, while MMT provides a deduction of the exact value of state variables and frequency fluctuations. A good correspondence between auto vibration system and its discrete model is stored and at the choice of 45 points using the difference scheme MMT.

IV. Analysis of Stability Condition

We will estimate the discretization error and conditions of stability by method (3). To this end, we will consider the test model of a conservative second-order system described by equation:

$$\frac{d^2 x}{dt^2} = -\omega^2 \cdot x, \quad (11)$$

where ω is the oscillation frequency. The harmonic signal with a single amplitude corresponds to the exact solution x_0 of equation (11). For analysis, it is more convenient to reduce (11) to the normal Cauchy form in the form of two first-order equations:

$$\frac{dx}{dt} = x^1; \qquad \frac{dx^1}{dt} = -\omega^2 \cdot x. \quad (11a)$$

After applying (3) to equations (11a), we arrive at the difference equations

$$x_{n+1} = x_n + \frac{2 \cdot h \cdot x^1_{n+1} \cdot x^1_n}{x^1_{n+1} + x^1_n} \quad i \quad x^1_{n+1} = x^1_n - \frac{2 \cdot h \cdot \omega^2 \cdot x_{n+1} \cdot x_n}{x_{n+1} + x_n}, \quad (11b)$$

the solution of which x_p approximated to solution (11) has the form:

$$\begin{bmatrix} x_{n+1} \\ x^1_{n+1} \end{bmatrix} = \mathbf{C} \cdot \boldsymbol{\rho} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \cdot \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix}^n$$

where **ρ** is the vector column of the discrete system (11b); **C** is a matrix of eigenvalues of eigenvectors of the system (11b). It is easy to make sure that the multipliers of the discrete model (11b) are determined by the eigenvalues of the continuous system (11). It is appropriate to analyze two first-order equations written with respect to eigenvalues (5) instead of the second-order system: to which the multipliers of the discrete system (11b) correspond

$\rho_1 = j \cdot \omega \cdot h + \sqrt{(1 - \omega^2 \cdot h^2)}$ and $\rho_2 = -j \cdot \omega \cdot h + \sqrt{(1 - \omega^2 \cdot h^2)}$.

Since the elements of the matrix **C** are determined by the initial conditions, which significantly affect the area of convergence of the computation process, and the error of discretization of the numerical method does not depend on them, it is sufficient to evaluate the modules and arguments of the multipliers:

$|\rho_1| = |\rho_2| = 1 + h^4 \cdot \omega^4$ and $\phi_1 = -\phi_2 = \mathrm{arctg}\,\frac{\omega \cdot h}{(1 - h^2 \cdot \omega^2 / 2)}$,

Thus, there is no error in determining the amplitude of oscillations up to members of smallness second order, when using method (3).

Given that the multiplier modules are close to a single circle, it can be argued that method (3) is stability. At the same time, it does not have the property A of stability, since the absolute magnitudes of the multipliers go beyond the frame of a single circle.

Having made similar calculations for the combinations of methods (4) - (7) we conclude that they all have the property A - stability, since

$$|\rho_1| - |\rho_2| \le= 1.$$

V. Results Numerical Experiments.

The results obtained by evaluating error for the obtained difference schemes (4) - (8) and (10) confirmed in modeling steady state and transient process in the generator Van der Pol [7], where was done modification and we introduced cubic nonlinear for the delay the transition process. How to confirm the results of computer experiments errors in determining state variables missing, and error calculation of the oscillation period fully consistent with the analytical expression (9) for different values of *k*. As example considered quartz generator circuit is shown in Fig. 1. For the calculation used for transistors model of Ebersa - Mola. The calculation was performed at the selecting of parameters the scheme such:

RX = 10 mohm; RQ = 20 ohm; R1 = 68 kohm; R2 = 20 ohm; R3 = 10 kohm;

R4 = 47 ohm; R5 = 1.2 kohm; CQ = 0,0010132118; C0 = 3 nF; C1 = 470 nF;

C2 = 470 nF; L = 1.0322536 Hz; E = 8 V.

For zero approximation state variables we used the calculation regimes DC:

X_1 = 1700; X_2 = 0.00168; X_3 = 4.56; X_4 = 0.324; X_5 = 3.67;

X_6 = 0.394; X_7 = 3.36; X_8 = 0.0025; X_9 = 4.25.

For this scheme, the vector of state variables presented in the form of

X (9) = {UCQ, JLQ, UC0, UC1, UCK2, UCE1, UCK2, UCE2, UC2}.

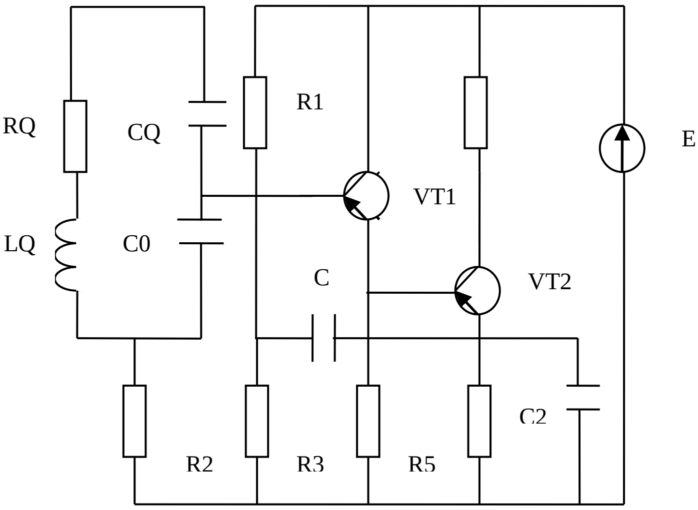

Figure 1. Circuit of quartz generator

The circuit quality factor is 1.5×10^6. The transition process is so protracted, that the required application acceleration algorithms [5-8, 13-14] to determine the steady state mode, because after the calculation of the first three periods is required by more than 20 iterations of Newton on every step of integration, resulting in significant time costs for the calculation. At the use of algorithms acceleration to determine steady state need the 13 extrapolations. This required 3-4 iterations of Newton in order to achieve steady state mode with accuracy 10-10. The analysis of transient process shows that the gains in calculation time are more than two orders of magnitude compared with the calculation without acceleration. The step calculating answered 209 points on the each period of calculation. By steady state regime received the following values of period and amplitude:

$$T = 2.0807823 \cdot 10^{-7}; \quad X_1 = 2 \cdot 10^{-9}; \quad X_2 = -0.00266651262;$$

$$X_3 = 1.7499755432; X_4 = 1.1051242378; X_5 = 6.8712539891;$$

$$X_6 = 0.5982940334; X_7 = 6.2893474619; \quad X_8 = 0.4881468158;$$

$$X_9 = 0.6422128153.$$

For estimation error in determining the characteristics of established regime, the step decreased twice and again performed numerical experiment. As follows from the results, the error in determining the oscillation period can be estimated by the formula:

$$\delta T = 3{,}3 \cdot \frac{h^2}{T^2}.$$

when using trapezoidal rule, which fully meets the previously given result and consistent with (9) with application of different combinations of order k for discretization.

Note that the results of calculation models crystal oscillator to within 10^{-12} coincide with the results given in [8]. In this aper extrapolation algorithm based on the implementation of procedures to automatically select the integration step, provided the convergence to steady mode is need 4 of extrapolation. But the advantage of the proposed procedure acceleration [6, 7] the lack of need for setting the oscillation period and a decrease on one order dimension matrix switching states, which significantly expands the scope of convergence.

VI. Conclusion

In the analysis of conservative systems in which no energy loss, the calculation of wireless devices with high quality factor, analysis of systems with long transient finding periodic modes in systems with complex dynamics advisable to use the discrete formula (10), which in the second order of smallness provides no error in determining both the amplitude and period of oscillations.

The conditions of stability proposed class of methods was obtain

Using the difference scheme (10) in the construction of discrete models in applications of computer analysis of electronic systems will increase their effectiveness and will ensure the reliability of their work.

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