Point spectrum as a profile of beliefs in conflict models

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Abstract – **This article examines the emergence of point spectra in discrete dynamical conflict systems and their role in belief formation within a society. By studying the structure and convergence of conflict trajectories to limit states, we analyze the conditions under which a discrete point spectrum emerges. A mathematical model is developed to describe the formation of individuals' beliefs through conflict interaction. The study focuses on systems where the number of possible beliefs is limited, examining both the aggregation of individual beliefs and the constraints that arise with multiple beliefs. The results show that while two beliefs can always be aggregated, the presence of three or more leads to more complex dynamics, where only a pair of individuals can be grouped by a common belief. The results provide insights into the formation of belief clusters in societies and offer a framework for understanding informational influence on conflict patterns.**

Keywords – **dynamical conflict system, conflict transformation (interaction), point spectrum, stochastic vector.**

Many mathematical models describe the process of beliefs formation and dissemination in society (for example, see [1]-[4]). Such models are very relevant for studying the dynamics of social networks, political systems, and other social structures where groups of individuals interact and gradually come to a common opinion. They allow modeling decision-making processes, including the influence of trust, limited confidence and mutual influence on the formation of collective opinion. These models are also applicable in areas such as the management of complex systems, economics and even in technologies where consistency between different components of the system is required.

For example, in [4], a mathematical model is proposed that describes the process of reaching consensus in systems with interacting agents. It examines the behavior of agents when making decisions in systems with two or more alternatives. Special cases of interaction between two agents and large groups are studied, and generalizations are given for cases with several alternatives. The key aspect is the gradual convergence of agent opinions through interaction, which leads to consensus.

This article proposes a method for constructing such models based on the concept of a conflict dynamic system and the interpretation of the phenomenon when a continuous spectrum is concentrated in a discrete spectrum. The study of the proposed belief model is carried out in terms of the structure of the point spectrum in time-limited events in discrete dynamic conflict systems of the form

$$\{\mu^t, \nu^t\} \overset{*}{\rightarrow} \{\mu^{t+1}, \nu^{t+1}\}, \qquad t = 0, 1, \dots,$$

where the probabilistic measures μ^t and ν^t are associated with stochastic matrices describing the states of a pair of alternative opponents, and $*$ denotes a non-commutative transformation that models conflict interaction between the opponents.

The spectral properties of the limiting measures $\mu^\infty = \lim_{t\to\infty} \mu^t$, $\nu^\infty = \lim_{t\to\infty} \nu^t$ have already been studied in several papers [5]-[14]. In particular, it has been established that, in the general case, the distributions of the limiting measures μ^∞ and ν^∞ are pure singularly continuous and have a fractal structure (see [5, 6, 10]). In [14], it was shown that the class of singularly continuous limiting distributions forms a set of full measure in the space of all limiting states of dynamical systems of this type. Point, particularly discrete, limiting spectrum can only arise in specific cases, with very rapid (exponential) local concentration of the approximating distributions μ^t and ν^t (see [11]).

The construction of an abstract model of a dynamical conflict system (see [13]) is determined by fixing three objects: Ω, $\mathcal{M}^{ss}(\Omega)$, $*$ specifically, the space Ω, the family of measures $\mathcal{M}^{ss}(\Omega)$ and the conflict transformation $*$. Here, Ω refers to any space in which the conflict between opponents unfolds. Physically, this repre sents a territory for the distribution of a certain resource. Mathematically, in the simplest case, Ω is the interval $[0,1]$ with the Lebesgue measure. It is important that the space Ω allows for the procedure of structural subdivision into subsets, as described further.

In the class of all structurally similar measures $\mathcal{M}^{ss}(\Omega)$ on the space Ω with a fixed primary measure λ we define three subclasses: $\mu \in \mathcal{M}_{\rm pp}$– measures with a purely point spectrum, $\mu \in \mathcal{M}_{\rm ac}$ – the measures are absolutely continuous with respect to λ, and the measures $\mu \in \mathcal{M}_{\rm sc}$ – are singularly continuous with respect to λ. Each structurally similar measure, $\mu \in \mathcal{M}^{ss}$, belongs to only one of the subclasses: $\mathcal{M}_{\rm ac}$, $\mathcal{M}_{\rm pp}$ or $\mathcal{M}_{\rm sc}$, which do not intersect.

At the initial moment of the discrete time $t = 0$, a pair of alternative opponents, denoted by A and B, choose strategies for distributing the statuses of their presence in the Ω space, as in a territory that is divided into separate regions:

$$\Omega = \bigcup_{i_0=1}^{n} \Omega_{i_0}, \qquad 2 \le n < \infty.$$

The starting strategies of the opponents are fixed by sets of numbers: $p_{i_0} \ge 0$ for A and $r_{i_0} \ge 0$ for B, the values of which characterize the statuses of the opponents in each of the regions Ω_{i_0} with weight λ_{i_0}. We assume that these values have the meaning of the probabilities of the presence of opponents A and B in Ω_{i_0}, i.e. $p_{i_0} := \mathbf{P}(\mathrm{A} \upharpoonright \Omega_{i_0})$, $r_{i_0} := \mathbf{P}(\mathrm{B} \upharpoonright \Omega_{i_0})$, where the symbol ↾ denotes the presence of the opponent in the region.

Therefore, at the moment $t = 0$, the strategy of each of the opponents is fixed by the corresponding stochastic vector,

$$\mathbf{p}^0 = (p_{01}, \ldots, p_{0i}, \ldots, p_{0n}), \mathbf{r}^0 = (r_{01}, \ldots, r_{0i}, \ldots, r_{0n}),$$
$$\sum_{i=1}^{\infty} p_{0i} = \sum_{i=1}^{\infty} r_{0i} = 1.$$

Next, we will assume that in the general case the coordinates $p_{i_0} \equiv p_{0i}$, $r_{i_0} \equiv r_{0i}$ are strictly positive and different:

$$p_i^0, r_i^0 > 0, p_i^0 \neq r_i^0, (\mathbf{p}^0, \mathbf{r}^0) = \sum_{i=1}^{n} p_i^0 r_i^0 \neq 1. \quad (3)$$

In the general case, the measures $\mu^t = \mu_{P^t}$, $\nu^t = \nu_{R^t}$, $t = 1, 2, \ldots$, which are constructed iteratively by $\mu_{P^{t-1}}$, $\nu_{R^{t-1}}$ according to this rule. The elements of the first $k \le t$ columns of the matrices P^t, R^t are given by the coordinates $p_{ik} \equiv p_i^k$, $r_{ik} \equiv r_i^k$, $i \in \overline{1, n}$ by vectors

$$\mathbf{p}^0, \mathbf{p}^1, \ldots, \mathbf{p}^k, \qquad \mathbf{r}^0, \mathbf{r}^1, \ldots, \mathbf{r}^k,$$

the coordinates of which are determined according to the formulas:

$$p_i^k \equiv p_{i_k k} = p_i^{k-1} \cdot \frac{1 - r_i^{k-1}}{1 - \theta^{k-1}},$$
$$r_i^k \equiv r_{i_k k} = r_i^{k-1} \cdot \frac{1 - p_i^{k-1}}{1 - \theta^{k-1}}, \quad (4)$$
$$\theta^{k-1} := \sum_{i=1}^{n} p_i^{k-1} r_i^{k-1}.$$

The above-described procedure of initial measures sets one of the possible transformations for describing successive acts of conflict interaction between opponents in discrete time. We denote this transformation by $*$. It generates trajectories of a dynamic conflict system,

$$\begin{Bmatrix} \mu^{t-1} \equiv \mu_{P^{t-1}} \\ \nu^{t-1} \equiv \nu_{P^{t-1}} \end{Bmatrix} \xrightarrow{*} \begin{Bmatrix} \mu^t \equiv \mu_{P^t} \\ \nu^t \equiv \nu_{P^t} \end{Bmatrix}, \qquad t = 1,2, \ldots \quad (7)$$

the states of which are fixed by a pair of measures associated with the corresponding stochastic matrices.

In the case $\mathbf{p}^0, \mathbf{r}^0 \in \mathbb{R}_{+,1}^{n=2}$ both measures μ^∞, ν^∞ associated to the limit matrices P^∞, R^∞ are purely point, $\mu^\infty, \nu^\infty \in \mathcal{M}_{\mathrm{pp}}$. In other case, when $\mathbf{p}^0, \mathbf{r}^0 \in \mathbb{R}_{+,1}^{n>2}$, one of the limit measures μ^∞, ν^∞ will be purely point, $\mu^\infty \in \mathcal{M}_{\mathrm{pp}}$, or $\nu^\infty \in \mathcal{M}_{\mathrm{pp}}$ if there exists only one index $1 \le \mathbf{i} \le \infty$ for vector $\mathbf{p}^0$, *or* $1 \le \mathbf{j} \le \infty$ for vector $\mathbf{r}^0$,such that one of the inequalities holds, respectively:

$$p_{\mathbf{i}}^0 > r_{\mathbf{i}}^0 \quad or \quad p_{\mathbf{j}}^0 < r_{\mathbf{j}}^0 \quad (8)$$

At the same time, if $\mu^\infty \in \mathcal{M}_{\mathrm{pp}}$, then $\nu^\infty \in \mathcal{M}_{\mathrm{sc}}$ and vice versa if $\nu^\infty \in \mathcal{M}_{\mathrm{pp}}$, then $\mu^\infty \in \mathcal{M}_{sc}$. If none of the inequalities (8) holds, then both limit measures are singularly continuous: $\mu^\infty, \nu^\infty \in \mathcal{M}_{\mathrm{pp}}$. In any case, the limit measures μ^∞, ν^∞ invariant with respect to the transformation $*$.

Let's consider a fixed family of measures $\mathcal{S} = \{\mu_\alpha \in \mathcal{M}^{ss}\}_{\alpha \in I}$ on the measurable space (Ω, Λ). The measures μ_α represent the states of individuals from an abstract society $\mathcal{S}$, identified by an index α. The size of the society is finite, $\#\{I\} = m < \infty$.

Under informational influence, the state of each individual changes in discrete time as $\mu_\alpha = \mu_\alpha^t$. The change law is given by the transformation $*$, which describes the conflict interaction with a fixed measure $\nu \in \mathcal{M}^{ss}$. The evolution of each individual's state over time occurs differently. If one of the limit measures has a point spectrum, $\mu_\alpha^{t=\infty} \in \mathcal{M}_{\mathrm{pp}}$, we interpret this as the formation of one of $1 \le \mathbf{i} \le n$ beliefs. The number of individuals who develop a certain belief depends on the strategy of informational influence, determined by the structure of ν.

At the initial moment each individual $\alpha \in I$ corresponds to a measure $\mu^0(\alpha)$ a associated with a matrix $P(\alpha)$ of type (1), where the first column is given by the coordinates of some stochastic vector $\mathbf{p}^0(\alpha) \in \mathbb{R}_{1,+}^n, n \ge 2$. The evolution of the states μ_α^t, $t = 0, 1, \ldots$, s well as the emergence of a point spectrum (i.e., the formation of a certain belief $\mathbf{i}$ among the individuals in society $\mathcal{S}$, is described by conflict interaction according to the rule (7), where the measure $\nu_{R^0} \in \mathcal{M}^{ss}$ is fixed by some vector $\mathbf{r}^0 \in \mathbb{R}_{1,+}^n$. This measure is associated with the source of informational influence. The vector $\mathbf{r}^0$ can naturally be associated with the strategy of informational influence since it determines the possibility of forming a certain belief $\mathbf{i}$.

According to analysis of the point spectrum structure, different limit distributions may arise for each of the measures $\mu_\alpha^t \in \mathcal{S}$. For $\mu_\alpha^\infty \in \mathcal{M}_{\mathrm{pp}}$, meaning that a specific belief $\mathbf{i}$ is formed in the corresponding individual, it is necessary and sufficient (see Theorem 4) that the following conditions hold in terms of the initial vector coordinates:

$$p_{\mathbf{i}}^0(\alpha) > r_{\mathbf{i}}^0, \quad p_i^0(\alpha) < r_i^0$$

Naturally, as before, to avoid additional explanations, we assume that all coordinates of the initial vectors $\mathbf{p}^0(\alpha)$ are non-zero.

Let $I_{\mathbf{i}}(\mathbf{r}^0)$ denote the subset of individuals for whom, under the influence of a fixed information source with strategy $\mathbf{r}^0$ belief $\mathbf{i}$ has been formed. The task is to investigate the dependence of the value $\#\{I_{\mathbf{i}}(\mathbf{r}^0)\}$ (the number of individuals with a fixed belief $1 \leq \mathbf{i} \leq n$) the number of individuals with a fixed belief $\mathbf{r}^0$, the strategy of informational influence.

Theorem 1. If $n = 2$, then for a family $\mathcal{S}$ of any arbitrary but finite cardinality $\#\{I\} = m < \infty$, there always exists a vector $\mathbf{r}^0$ such that $\#\{I_\mathbf{i}(\mathbf{r}^0)\} = m$ for both beliefs $\mathbf{i} = 1, 2$.

In the case where there are only two beliefs among all individuals in an abstract society, it is possible to form a single belief by choosing an appropriate strategy of informational influence. However, when there are three possible beliefs, i.e., when $n = 3$, a similar result does not hold, even for a society of three individuals. Nevertheless, any two arbitrary individuals from a finite society can always be united by one of the three possible beliefs.

Theorem 2. Let $n = 3$, $\#\{I\} = m < \infty$ for any pair of vectors $\mathbf{p}^0(\alpha), \mathbf{p}^0(\beta)$, there exists a vector $\mathbf{r}^0$ such that conditions (8) are simultaneously satisfied for α and β for both beliefs $1 \leq \mathbf{i} \leq 3$.

Let us denote by $\mathcal{S}_i$ the subset of all related individuals. For a fixed belief $1 \leq \mathbf{i} \leq n$ individuals $\alpha, \beta \in I$ are called related if the following conditions hold:

$$p_i^0(\alpha), p_i^0(\beta) < \frac{1}{n}, \quad \forall\, i \neq \mathbf{i}.$$

When there are more than three possible beliefs, it is generally impossible to unite any pair of individuals by a common belief. However, if the number of beliefs is significantly smaller than the number of individuals, clusters of individuals close to a certain belief inevitably arise in the vector space corresponding to the individuals from $\mathcal{S}$, since their number exceeds the number of possible beliefs. The following theorem states that all related individuals can be united by a single common belief.

Theorem 3. For each set $\mathcal{S}_i$ there exists a vector $\mathbf{r}^0$ such that conditions (8) are fulfilled simultaneously for all $\alpha \in \mathcal{S}_i$, therefore, all measures with $\mu^\infty(\alpha)$, $\alpha \in \mathcal{S}_i$ have the same spectral profile.

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