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On the Solution of a Linear Fuzzy Cauchy Problem with Constant Coefficients

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Abstract— **In this report we consider a linear fuzzy Cauchy problem with constant coefficients and present an analytical formula for its solution.**

Keywords—fuzzy; Cauchy problem; Hukuhara derivative; linear

I. Introduction

The theory of fuzzy sets, introduced by L.A. Zadeh [1], allows us to model non-probabilistic uncertainties with ease. This fact explains the growing interest in both the theoretical and practical aspects of fuzzy set theory in recent years. Applications of fuzzy set theory can be found in many fields of science, including the physical, mathematical, and engineering sciences [2–9]. More recently, significant progress has been made in the theory of fuzzy differential equations, differential inclusions with fuzzy right-hand sides, and fuzzy differential inclusions, as well as in the theory of controlled fuzzy differential equations and controlled fuzzy differential inclusions (see [5–14] and the references therein).

The report examines a linear fuzzy differential equation with constant coefficients and provides an analytical formula for its solution.

II. Preliminaries

Let R be the set of real numbers and R^n be the n-dimensional Euclidean space ($n \ge 2$). Denote by $conv(R^n)$ the set of nonempty compact and convex subsets of R^n.

For two given sets $X, Y \in conv(R^n)$ and $\lambda \in R$, the Minkowski sum and scalar multiple are defined by $X + Y = \{x + y \mid x \in X, y \in Y\}$ and $\lambda X = \{\lambda x \mid x \in X\}$.

Also, let's add one more operation: the product of a matrix with a set $AX = \bigcup_{x \in X} Ax$, where $A \in R^{n\times n}$ is real matrix of size $n \times n$ and $X \in conv\left(R^n\right)$.

Consider the Pompeiu-Hausdorff distance $h(\cdot,\cdot)$ given by
$h(X,Y) = \min\left\{r \ge 0 \mid X \subset Y + B_r(0),\ Y \subset X + B_r(0)\right\}$,

where $B_r(0) = \left\{x \in R^n \mid \|x\| \le r\right\}$ is the closed ball with radius r centered at the origin ($\|x\|$ denotes the Euclidean norm).

It is known that $(conv(R^n), h)$ is a complete metric space [7].

Let E^n be a family of all $u : R^n \to [0,1]$ such that u satisfies the following conditions:

1) u is normal, i.e. there exists $x_0 \in R^n$ such that $u(x_0) = 1$;
2) u is fuzzy convex, i.e. for any $x, y \in R^n$ and $0 \le \lambda \le 1$ $u\left(\lambda x + (1-\lambda) y\right) \ge \min\left\{u(x), u(y)\right\}$;
3) u is upper semicontinuous, i.e. for any $x_0 \in R^n$ and $\varepsilon > 0$ there exists $\delta(x_0, \varepsilon) > 0$ such that $u(x) < u(x_0) + \varepsilon$ whenever $\| x - x_0 \| < \delta(x_0, \varepsilon)$, $x \in R^n$;
4) the closure of the set $\left\{x \in R^n : u(x) > 0\right\}$ is compact.

If $u \in E^n$, then u is called a fuzzy set, and E^n is said to be a space of fuzzy sets.

Definition 1.[4] The set $\left\{x \in R^n : u(x) \ge \alpha\right\}$ is called the α *-level* $[u]^\alpha$ of a fuzzy set $u \in E^n$ for $0 < \alpha \le 1$. The closure of the set $\left\{x \in R^n : u(x) > 0\right\}$ is called the 0-level $[u]^0$ of a fuzzy set $u \in E^n$.

Theorem 1. [5] (Stacking Theorem). If $u \in E^n$ then
1) $[u]^\alpha \in conv(R^n)$ for all $\alpha \in [0,1]$;
2) $[u]^{\alpha_2} \subset [u]^{\alpha_1}$ for all $0 \le \alpha_1 \le \alpha_2 \le 1$;
3) if $\{\alpha_k\}$ is a nondecreasing sequence converging to $\alpha > 0$, then $[u]^\alpha = \bigcap_{k \ge 1} [u]^{\alpha_k}$.

Conversely, if $\{X_\alpha : \alpha \in [0,1]\}$ is the family of subsets of R^n satisfying conditions 1) - 3) then there exists $u \in E^n$ such that $[u]^\alpha = X_\alpha$ for $0 < \alpha \le 1$ and $[u]^0 = \overline{\bigcup_{0 < \alpha \le 1} X_\alpha} \subset X_0$.

We define $u+v$ and Au by $\left[u+v\right]^{\alpha}=\left[u\right]^{\alpha}+\left[v\right]^{\alpha}$ and $\left[Au\right]^{\alpha}=A\left[u\right]^{\alpha}$ respectively, for every $\alpha\in[0,1]$ and $A\in R^{n\times n}$

Define $D:E^n\times E^n\to[0,\infty)$ by the relation

$$D(u,v)=\sup_{0\le\alpha\le1} h\left(\left[u\right]^{\alpha},\left[v\right]^{\alpha}\right).$$

Then D is a metric in E^n. Further we know that [5,7]:

i) $\left(E^n,D\right)$ is a complete metric space,

ii) $D\left(u+w,v+w\right)=D\left(u,v\right)$ for all $u,v,w\in E^n$,

iii) $D\left(\lambda u,\lambda v\right)=\left|\lambda\right|D\left(u,v\right)$ for all $u,v\in E^n$ and $\lambda\in R$.

Let X,Y,Z be in $conv(R^n)$. The set Z is the Hukuhara difference of X and Y, if $Y+Z=X$, i.e. $Z=X\overset{H}{-}Y$.

Definition 2.[5,7] A mapping $F:[0,T]\to conv(R^n)$ is differentiable in the sense of Hukuhara at $t\in[0,T]$ if for some $h>0$ the Hukuhara differences $F(t+\Delta)\overset{H}{-}F(t)$, $F(t)\overset{H}{-}F(t-\Delta)$ exist in $conv(R^n)$ for all $0<\Delta<h$ and there exists $D_H F(t)\in conv(R^n)$ such that

$$\lim_{\Delta t\to0} h\left(\Delta^{-1}(F(t+\Delta)\overset{H}{-}F(t)),D_H F(t)\right)=0$$

and

$$\lim_{\Delta t\to0} h\left(\Delta^{-1}(F(t)\overset{H}{-}F(t-\Delta)),D_H F(t)\right)=0.$$

Here $D_H F(t)$ is called the Hukuhara derivative of $F(\cdot)$ at t.

Definition 3.[5,7] A mapping $x:[0,T]\to E^n$ is called differentiable at $t\in[0,T]$ if for any $\alpha\in[0,1]$ the set-valued mapping $x_\alpha(t)=\left[x(t)\right]^{\alpha}$ is differentiable in the sense of Hukuhara at point t with $D_H x_\alpha(t)$ and the family $\left\{D_H x_\alpha(t):\alpha\in[0,1]\right\}$ defines a fuzzy set $D_H x(t)\in E^n$.

If $x:[0,T]\to E^n$ is differentiable at $t\in[0,T]$, then we say that $D_H x(t)$ is the fuzzy derivative of $x(\cdot)$ at the point $t\in[0,T]$.

III. Linear Fuzzy Problem

Now, consider the Cauchy problem

$$D_H x(t)=A\,x(t)+f,\quad x(0)=x_0, \tag{1}$$

where $A\in R^{n\times n}$, $x_0,f\in E^n$.

The fuzzy mapping $x(\cdot)$ will be called the solution of the system (1) on the interval $[0,T]$ if it is continuously and satisfies system (1) on $[0,T]$.

Theorem 2. If the matrix A is non-degenerate, then system (1) has a unique solution of the form

$$x\left(t\right)=x_0+\sum_{k=1}^{\infty}\left[\frac{A^k t^k}{k!}\left(x_0+A^{-1}f\right)\right]$$

for all $t\in[0,T]$.

Remark. If the singular values of the matrix A are such that $\sigma_1=...=\sigma_n=\sigma$ and $Ax_0=\sigma x_0$, then

$$x\left(t\right)=e^{\sigma t}x_0+\sum_{k=1}^{\infty}\left[\frac{A^k t^k}{k!}A^{-1}f\right].$$

Remark. If also $Af=\sigma f$, then

$$x\left(t\right)=e^{\sigma t}x_0+\frac{1}{\sigma}\left(e^{\sigma t}f-f\right)$$

where $\left[g-f\right]^{\alpha}=\left[g\right]^{\alpha}\overset{H}{-}\left[f\right]^{\alpha}$ for all $\alpha\in\left[0,1\right]$.

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