Integral Conjugation Condition for Smart Geobarriers with Assimilation of Sensor Data

https://doi.org/10.31713/MCIT.2025.062

Olena Belozerova
National University of Water and Environmental Engineering
Rivne, Ukraine
ol.d.kozhushko@nuwm.edu.ua

Petro Martyniuk
National University of Water and Environmental Engineering
Rivne, Ukraine
p.m.martyniuk@nuwm.edu.ua

Abstract— **The long-term integrity of geoenvironmental containment systems is critical for preventing groundwater contamination, while conventional monitoring is only reactive. This work addresses the need for proactive, predictive monitoring by deriving a novel integral conjugation boundary condition for a smart geobarrier. This condition incorporates real-time sensor data through a Newtonian nudging data assimilation term, providing a mathematically rigorous mechanism to continuously steer a physical model toward observed reality. As an example of application, a boundary value problem for the process of filtrational consolidation is formulated.**

Keywords—mathematical modeling, filtrational consolidation, conjugation condition, smart geobarrier, data assimilation.

I. Introduction

The management of waste from municipal, industrial, and mining activities presents a significant environmental engineering challenge. Critical infrastructure like landfills and wastewater lagoons pose a substantial threat to groundwater resources. The primary defense is the geoenvironmental containment system, traditionally relying on passive barriers like compacted clay liners and high-density polyethylene (HDPE) geomembranes.

However, the long-term integrity of these passive systems is uncertain, as they are susceptible to degradation and defects [1]. Conventional monitoring is reactive, confirming failures only after contamination has occurred, at which point remediation is costly and often ineffective. In the long term, this approach cannot provide safe and sustainable solution.

The recently introduced "smart" or "intelligent" geobarriers have embedded sensing capabilities, allowing for continuous, real-time monitoring. They perform their traditional functions of separation, filtration, drainage, reinforcement, and containment, while simultaneously acting as integrated sensor networks. This dual functionality enables them to transmit critical information about the system's physical state, such as strain, stress, and temperature. This approach avoids the need for discrete, intrusive instrumentation [2].

Sensing capability of smart geobarriers is often achieved by embedding electrically conductive materials, that would change properties drastically if any leak in the geobarrier occurs. More advanced technology, called fiber-optic sensing, uses Fiber Bragg Gratings (FBG) – periodic variations in the refractive index of an optical fiber. Such optical sensors reflect light at a specific wavelength, which is highly sensitive to mechanical strain and temperature. A key advantage is multiplexing, where hundreds of FBG sensors can be placed along a single fiber for quasi-distributed sensing [3]. The fiber-optics sensors can be applied to detect strain, water pressure and even concentration of chemicals [4].

Moreover, the presence of multiple sensors allows for more precise monitoring and application of this data to predictive modeling. Together with adequate mathematical models and data assimilation techniques, they provide a framework to merge real-time sensor data with model predictions, creating a dynamically updated understanding of the system's state.

However, accurate model representation of this geobarriers is critical for further implementation of such intelligent systems. As a kind of thin inclusion, smart geobarrier would require complex conjugation conditions for modeling, similar to the conditions proposed in [5, 6]. Further incorporation of data assimilation term into this conjugation conditions would ensure the accurate and practical implementation of sensor data in the mathematical model.

II. Newtonian Nudging Assimilation

Data Assimilation (DA) is a set of mathematical techniques designed to produce an optimal estimate of the state of a dynamic system by combining information from a numerical model and observations. The numerical model provides a physically consistent description of the system, but it is imperfect due to errors in its initial conditions, parameters, and physical approximations. Observations from sensors provide direct information about the real state of the system, but they are typically sparse, indirect, and contain measurement errors. The purpose of DA is to merge these two incomplete and uncertain sources of information, decreasing uncertainty and improving the model's forecasting capability [7].

The use of data assimilation is especially advantageous for refining the conjugation conditions for geobarriers. It would allow to update the model with accurate sensor information, while sustaining the continuity of model variable fields and accounting for possible measurement flaws.

The method we chose for this problem, called Newtonian nudging or Newtonian relaxation, is a continuous data assimilation method that gently steers, or "nudges," a model's simulation toward observations as it runs. Instead of solving a large-scale optimization problem like variational methods, nudging adds a non-physical relaxation term directly into the model's prognostic equations [8]. In general terms, its implementation can be described with the following equation:

$$\frac{\partial u}{\partial t} = F(u, X, t) + G \cdot W(X, t) \cdot (u_{obs} - u), \qquad (1)$$

where $F(u, X, t)$ represents the original model physics; $(u_{obs} - u)$ is the innovation, or the difference between the observation and the model's current state; G is the nudging coefficient, which determines the strength of the correction and represents a relaxation time scale; and $W(X, t)$ represents weighting functions that localize the nudging's influence in space and time [9].

At each time step, the nudging term applies a correction proportional to the difference between the model state and the observations, continuously pushing the simulation closer to reality. This approach is computationally much simpler than variational methods, as it does not require the development of a complex adjoint model.

III. Integral conjugation condition with data assimilation term

For derivation of this condition, we consider a low-permeability geobarrier ω of small thickness d, installed in the soil medium. The geobarrier has sensing capabilities that allow to observe strain, which can be used to estimate soil water pressure head h_{obs}. The water transfer inside the geobarrier can be described with the equation of elastic filtration. With addition of the Newtonian nudging term, the governing equation is as follows:

$$\beta_\omega(t)\frac{\partial h}{\partial t} = \frac{\partial}{\partial \xi}\left(k_\omega(t)\frac{\partial h}{\partial \xi}\right) - \\ -GW(\xi, t)(h_{obs} - h), 0 < \xi < d, t > 0, \qquad (2)$$

$$h(0, t) = h^-(t), h(d, t) = h^+(t), t > 0. \qquad (3)$$

Here, h is pressure head, h^+ and h^- are pressure head values on the boundaries of the inclusion, $k_\omega(t)$ is the barrier hydraulic conductivity, $\beta_\omega(t) = \gamma a_\omega/(1 + e_\omega)$, where γ is specific weight of the pore water, a_ω is compressivity of the porous geobarrier material, $e_\omega = n_\omega/(1 - n_\omega)$ is porosity coefficient, n_ω is porosity of the geobarrier. For simplicity, we consider a one-dimensional problem here, but the same can be applied to the higher-dimensional cases.

Integrating the equation (2), we get

$$k_\omega(t)\frac{\partial h}{\partial \xi} = \beta_\omega(t)\int_0^\xi \frac{\partial h}{\partial t}dz - \\ -\int_0^\xi GW(z, t)(h_{obs} - h)dz + h_1,$$

where h_1 is a unknown function that may depend on time t. Than

$$h(\xi, t) = \frac{\beta_\omega(t)}{k_\omega(t)}\int_0^\xi \int_0^\varsigma \frac{\partial h}{\partial t}dz d\varsigma - \\ -\frac{1}{k_\omega(t)}\int_0^\xi \int_0^\varsigma GW(z, t)(h_{obs} - h)\, dz d\varsigma + \\ +\frac{\xi}{k_\omega(t)}h_1(t) + h_2(t), \qquad (4)$$

where h_2 is also an unknown function that only depends on time. Further, from (4) and boundary conditions (3) we have

$$h(0, t) = h_2 = h^-, \\ h(d, t) = \frac{\beta_\omega(t)}{k_\omega(t)}\int_0^d \int_0^\varsigma \frac{\partial h(z, t)}{\partial t}dz d\varsigma - \\ -\frac{1}{k_\omega(t)}\int_0^\xi \int_0^\varsigma GW(h_{obs} - h)\, dz d\varsigma + \\ +\frac{d}{k_\omega(t)}h_1(t) + h_2(t) = h^+. \qquad (5)$$

For the system of equations (5), we have

$$h_1 = \frac{k_\omega(t)}{d}\left[h^+ - h^- - \frac{\beta_\omega(t)}{k_\omega(t)}\int_0^d \int_0^\varsigma \frac{\partial h(z, t)}{\partial t}dz d\varsigma + \\ +\frac{1}{k_\omega(t)}\int_0^d \int_0^\varsigma GW(h_{obs} - h)\, dz d\varsigma\right].$$

The difference $(h^+ - h^-)$, further denoted $[h]$, is the pressure head jump that occurs when the water is passing through the inclusion.

Applying this to the the problem (1), (2) we have

$$h(\xi, t) = h^- + \frac{\beta_\omega(t)}{k_\omega(t)}\int_0^\xi \int_0^\varsigma \frac{\partial h}{\partial t}dz d\varsigma - \\ -\frac{1}{k_\omega(t)}\int_0^\xi \int_0^\varsigma GW(\varsigma, t)(h_{obs} - h)\, dz d\varsigma + \\ +\frac{\xi}{k_\omega(t)}h_1.$$

For the final derivation of the conjugation condition, we need to use the soil flux rate u. According to Darcy's law, we have

$$u = -k_\omega(t)\frac{\partial h}{\partial \xi} = -\beta_\omega(t)\int_0^\xi \frac{\partial h}{\partial t}dz + \\ +\int_0^\xi GW(h_{obs} - h)dz - h_1. \qquad (6)$$

Thus, we get the following conjugation condition for non-ideal contact with a nudging term:

$$u^- = u|_{\xi=0} = -\frac{k_\omega(t)}{d}\left([h] - \frac{\beta_\omega(t)}{k_\omega(t)} \times \\ \times \int_0^d \int_0^\varsigma \frac{\partial h(z, t)}{\partial t}dz d\varsigma + \right. \qquad (7)$$

$$+\frac{1}{k_\omega(t)}\int_0^d\int_0^\varsigma GW(h_{obs}-h)\,dzd\varsigma\Big);$$

$$\begin{aligned}u^+ = u|_{\xi=d} = -\beta_\omega(t)\int_0^d \frac{\partial h}{\partial t}dz + \\ +\int_0^d GW(z,t)(h_{obs}-h)dz - \\ -\frac{k_\omega(t)}{d}\Big([h] - \frac{\beta_\omega(t)}{k_\omega(t)}\int_0^d\int_0^\varsigma \frac{\partial h(z,t)}{\partial t}dzd\varsigma + \\ +\frac{1}{k_\omega(t)}\int_0^d\int_0^\varsigma GW(h_{obs}-h)\,dzd\varsigma\Big).\end{aligned} \quad (8)$$

IV. Application to a boundary value problem

The conjugation conditions (7), (8) can be used in the mathematical model to simulate the processes in the soil with smart geobarrier installed. For example, consider a soil layer of thickness l with a geobarrier ω of thickness d, $d \ll l$, located on depth $x=\xi$, where material of the thin inclusion ω has characteristics different from that of the soil medium. Applying the filtrational consolidation equation according to [10], we get the following boundary value problem:

$$\frac{\gamma a}{1+e}\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\Big(k(t)\frac{\partial h}{\partial x}\Big), x \in \Omega_1 \cup \Omega_2, t>0, \quad (9)$$

$$h(x,t)|_{x=0} = 0, u(x,t)|_{x=l} = 0, t>0, \quad (10)$$

$$h(x,0) = h_0(x), x \in \overline{\Omega_1} \cup \overline{\Omega_2}, \quad (11)$$

$$\begin{aligned}u^- = u|_{x=\xi-0} = -\frac{k_\omega(t)}{d}\Big([h] - \frac{\beta_\omega(t)}{k_\omega(t)}\times \\ \times\int_0^d\int_0^\varsigma \frac{\partial h(z,t)}{\partial t}dzd\varsigma + \\ +\frac{1}{k_\omega(t)}\int_0^d\int_0^\varsigma GW(h_{obs}-h)\,dzd\varsigma\Big),\end{aligned} \quad (12)$$

$$\begin{aligned}u^+ = u|_{x=\xi+0} = -\beta_\omega(t)\int_0^d \frac{\partial h}{\partial t}dz + \\ +\int_0^d GW(z,t)(h_{obs}-h)dz - \\ -\frac{k_\omega(t)}{d}\Big([h] - \frac{\beta_\omega(t)}{k_\omega(t)}\int_0^d\int_0^\varsigma \frac{\partial h(z,t)}{\partial t}dzd\varsigma + \\ +\frac{1}{k_\omega(t)}\int_0^d\int_0^\varsigma GW(h_{obs}-h)\,dzd\varsigma\Big).\end{aligned} \quad (13)$$

In the formulation above, $\Omega_1 = (0;\xi), \Omega_2 = (\xi;l)$, $0<\xi<l$, $h_0(x)$ is a set function, γ is specific weight of the pore water, a is soil compressivity coefficient, $e = n/(1-n)$ is soil porosity coefficient, n is soil porosity, k and k_ω are hydraulic conductivities of the soil and the geobarrier, respectively, u is soil flux rate determined according to (6), u^+ and u^- are water flux rates at $x=\xi-0$ and $x=\xi+0$, respectively.

V. Conclusion

In this study, we proposed a theoretical framework for integrating real-time sensor data into the mathematical modeling of smart geobarrier systems. The derived integral conjugation boundary condition augmented with a Newtonian nudging term provides a direct and physically consistent method for assimilating observational data, allowing a numerical model to be continuously corrected by measurements. The utility of this condition was demonstrated by formulating a boundary value problem for filtrational consolidation.

Subsequent work should focus on numerical validation through synthetic experiments, conducting a parametric analysis of the nudging coefficient. This foundational research provides an important step toward creating a "digital twin" for smart geobarriers, bridging the gap between advanced sensor technology and predictive mathematical modeling.

VI. References

[1] J. Atmadja and A. Bagtzoglou, Amvrossios, “State of the art report on mathematical methods for groundwater pollution source identification,” Environmental Forensics, vol. 2, pp. 205–214, 2001.

[2] M. Abedi, R. Fangueiro, A.G. Correia, and J. Shayanfar, “Smart geosynthetics and prospects for civil infrastructure monitoring: a comprehensive and critical review,” Sustainability, vol. 15, no. 12, 9258, 2023.

[3] H. Pei, Y. Zhong, W. Feng, and H. Zhu, “A novel effective stress sensor based on FBG sensing technology for effective stress measurement in soil,” Measurement, vol. 243, no. 15, 116331, February 2025.

[4] H. Dong, S. J. Wei, L. Ding, R. Yao, J. Huang, W. Xu et al., “Monitoring chloride ion concentration in saturated cohesive soils by optical fiber technology,” Measurement, vol. 253, part B, 117569, 2025.

[5] P. Martyniuk, O. Ulianchuk-Martyniuk, V. Herus, O. Pryshchepa, “Integral conjugation conditions for a discontinuous filtration flow via a geobarrer in the case of its biocolmatation,” International Journal of Applied Mathematics, vol. 37, no. 1, pp. 21–28, 2024.

[6] O. Stepanchenko, P. Martyniuk, O. Belozerova, L. Shostak, “Balance boundary conditions for thin boundary layers: derivation from the integral conjugation conditions,” International Journal of Applied Mathematics, vol. 36, no. 1, pp. 107–116, 2023.

[7] H. Moradkhani, “Hydrologic remote sensing and land surface data assimilation,” Sensors, vol. 8, no. 5, pp. 2986-3004, May 2008.

[8] C. Paniconi, M. Marrocu, M. Putti, and M. Verbunt, “Newtonian nudging for a Richards equation-based distributed hydrological model,” Advances in Water Resources, vol. 26, pp. 161–178, 2003.

[9] O.D. Kozhushko, M. V. Boiko, M. Yu. Kovbasa, P. M. Martyniuk, O. M. Stepanchenko, and N. V. Uvarov, “Field scale computer modeling of soil moisture with dynamic nudging assimilation algorithm”, Mathematical Modeling and Computing, vol. 9, no. 2, pp. 203–216, 2022.

[10] O. R. Michuta, and P. M. Martyniuk, “Nonlinear Evolutionary Problem Of Filtration Consolidation With The Non-Classical Conjugation Condition,” Journal of Optimization Differential Equations and their Applications, vol. 30, no.1, pp. 71–87, 2022.